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EXISTENCE AND DETERMINATION OF SUPER-HARMONIC SYNCHRONIZATIONS AS SOLUTIONS OF A QUASI-LINEAR PHYSICAL SYSTEM

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Existence of super-harmonic synchronizations of orders 2, 3, and 4 of a quasi-linear physical system described by the quasi-linear differential equations :

$$\ddot{x} + k_1 x + k_2 f(\Omega t) x^n = 0, \quad k_2 \ll 1,$$

for an odd integer n (3 and 5) and a continuous periodic function $f(\Omega t)$ (sinusoidal function) are proved. Also those synchronizations are determined in the plane k_1, k_2 for a given initial point $x_0, \dot{x}_0$.

1. INTRODUCTION

Many phenomena of the non-linear physical systems with external excitation are well known, but until now little attention has been given to the phenomena of super, or sub-super, or super-sub harmonic synchronizations. "These derive their names from the relationship between the frequencies of the synchronizing force, and the system", the frequency of the system being respectively integral multiple, or rational fraction of the synchronizing force.

Gauchey[10] in his work about a system described by Duffing's equation, proved the existence of ultra-harmonic motion by using the linearization method. Also Tomas[7] proved the existence of ultra-sub harmonic resonance for quasi-linear systems, by using harmonic balance method. Ludke[4] discussed the mathematical and physical conditions under which these types of resonance occur. Levenson[9] proved the existence of ultra-sub harmonic resonance, by using a numerical method. Atkinson[5] proved the existence of super-harmonic solution by using the electronic differential analyzer. Joseph and Ibrahim[10] studied in detail the existence of such oscillations modes for Duffing's equation by using the perturbation procedure.

Generally non-linear systems possess the distinctive characteristic that various types of synchronizations may exist, depending on the initial conditions in the system and on the relative values between the natural frequency and the frequency of synchronizing force. We have proved[1,2] the existence of the phenomena of harmonic and sub-harmonic synchronizations of even and odd order.

This paper is to focus attention on the existence of super-harmonic synchronizations of order m (= 2, 3, 4) and localize it, in the plane k_1, k_2, for a given initial point, for a quasi-linear physical system, characterized by the quasi-linear differential equation :

$$\ddot{x} + k_1 x + k_2 f(\Omega t) x^n = 0, \quad k_2 \ll 1 \qquad ...(1)$$

in the case of odd synchronizing force (n = 3, 5) and $f(\Omega t)$ is continuous periodic function, sinusodial function, by using the index method[3] which enables, us to prove the existence of such synchronizations and localize those synchronizations in the plane k_1, k_2, for for given initial point x_0, $\dot{x}_0$.

2. Index Method

Putting $\frac{dx}{dt} = y$, in eqn. (1), we get the following two dimensional system :

$$\frac{dx}{dt} = y$$

$$\frac{dy}{dt} = -k_1 x - k_2 x^n f(\Omega t). \qquad ...(2)$$

For given initial point $(x_0, y_0, k_1, k_2, t_0)$ we can associate with the system (2), th following :

(i) *The indicatrix* $C(x_0, y_0, t_0, k_1, k_2)$ — It is a closed curve, which lies in the phase plane (x, y) consisting of the points

$$(x(t_0 + T, t_0, x_0, y_0, k_1, k_2), y(t_0 + T, t_0, x_0, y_0, k_1, k_2))$$

when t_0 covers the interval $[0, T_0]$, where T, T_0 are the periods of the oscillations, and the system (2), respectively.

(ii) *The exceptional point* — If the point $(x_0, \dot{x}_0, t_0, k_1, k_2)$ lies on its indicatrix it is called the exceptional point, from which issues at least one periodic oscillation with period minimal T. If $T = \frac{T_0}{m}$, where m is an integer $(m \neq 1)$ we have what is called super harmonic oscillations of order m.

(iii) *The index* — If the point $(x_0, y_0, t_0, k_1, k_2)$ does not lie on its indicatrix $C(x_0, y_0, t_0, k_1, k_2)$, we define a number $i \in Z$, called the index associated to this point. It is defined by the number of rotations of the vector, with origin $(x_0, y_0, t_0, k_1, k_2)$, and the end $[x(t_0+T, t_0, x_0, y_0, k_1, k_2)\ y(t_0+T, t_0, x_0, y_0, k_1, k_2), t_0, k_1, k_2]$, when t_0 covers the interval $[0, T_0]$.

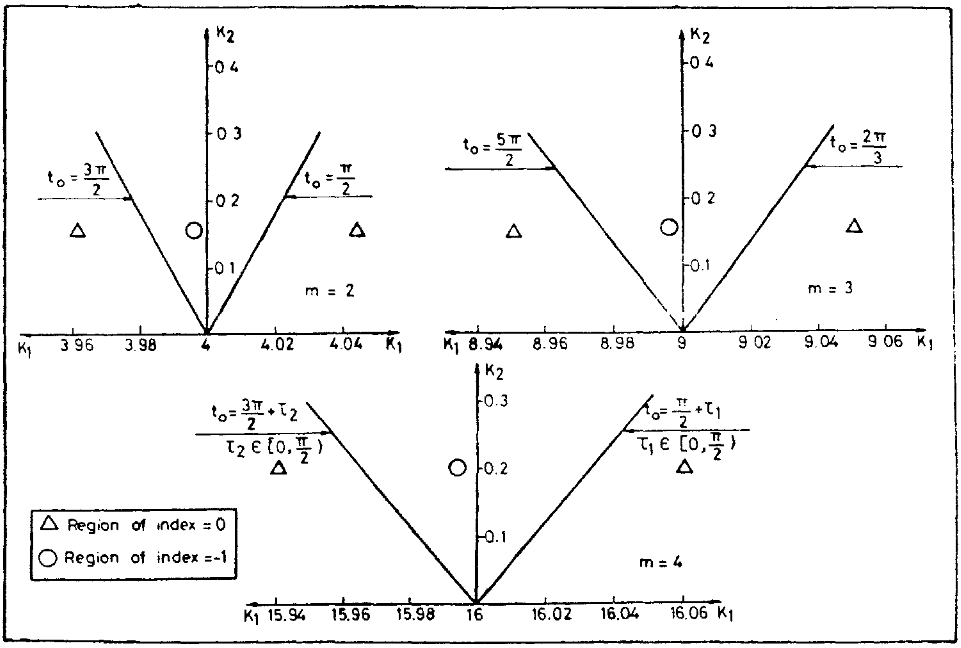

Fig. 1. Super-Harmonic Synchronizations of order m (m = 2, 3, 4) at (X) t_0=0.5, ($\dot{X}$) t_0 = 0,, n = 3.

If for two different points $((x_0, y_0, t_0, k_1, k_2), (x_0, y_0, t_0, k_1', k_2'))$ the indices are defined and differents then on all continuous curves in the plane (k_1, k_2) joining the two points (k_1, k_2), (k_1', k_2') there exists at least one exceptional point.

(iv) *The separatrix* — It is a curve which lies in the plane k_1, k_2. It consists of all exceptional points and divides the plane into two regions where the indices are defined.

3. Super-Harmonic Synchronizations

By applying the preceding method using Runge-kutta method for solving second order differential equation, we observe that : for k_2=0, k_1=m, where m=2, 3 and 4, the associated indicatrix, to the initial point x_0, $\dot{x}_0$ reduces to this point. But for $k_2 \neq 0$, the associated indicatrix appears as a closed simple curve, making one revolution about the initial point x_0, $\dot{x}_0$, when t_0 varies over the interval $[0, 2\pi]$. Then we deduce that the index is defined and equal to -1. If we take k_1 in the neighbourhood of m^2, we observe that the indicatrix is displaced passing by the initial point twice, one for $k_1 > m^2$ and the other for $k_1 < m^2$. Then we deduce that the index changes from -1 to 0 twice, which proves the existence of two exceptional points.

By determining the values of k_1 at which the initial point x_0, $\dot{x}$, lies on its indicatrix we obtain the associated exceptional points.

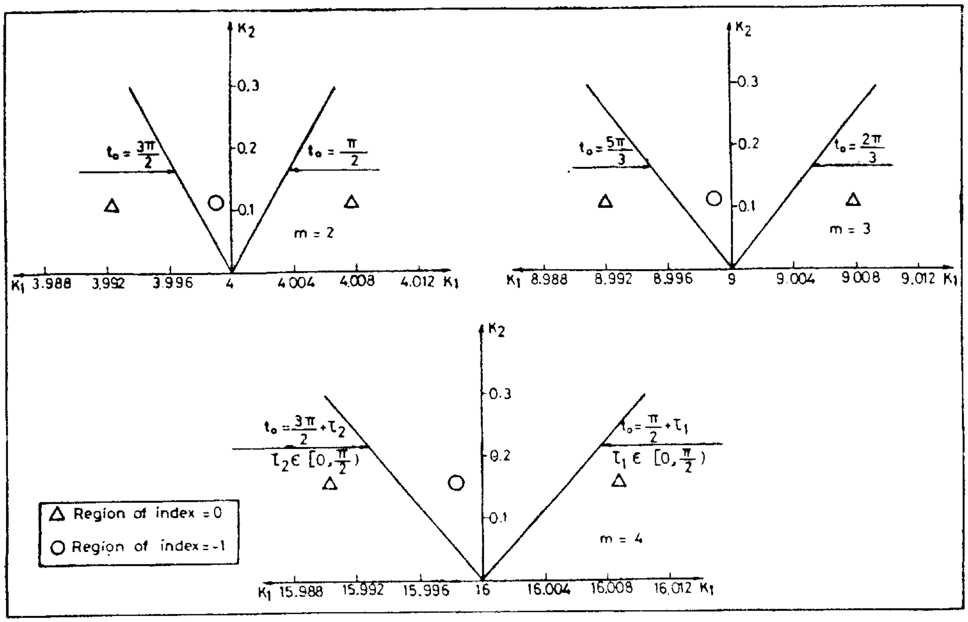

FIG. 2. Super-Hormonic Synchronizations of order m (m = 2, 3, 4) at $(X)\ t_0 = 0.5$, $(\dot{X})\ t_0 = 0$, $n = 5$.

By varying the parameter k_2, over the axis $k_1=0$, and determining the values of k_1 corresponding to the exceptional points, we obtain the set of exceptional points "Separatrices". Figures 1 and 2 represent the obtained set of exceptional points, for $n=3$ and 5 respectively, at $x_0 = 0.5$, $\dot{x}_0 = 0$, so that from those results we deduce that the physical system described by equation (1) admits as super-harmonic synchronization of order m (m = 2, 3 and 4) two synchronizations one for $k_1 < m^2$. Also it is clear from the figures that the relation between k_1 and k_2 is linear, for $k_2 \ll 1$.

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